Structured De-Chirp for Compressive Sampling of LFM Waveforms

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***Abstract*—Stretch processing is often used by radar systems to capture LFM waveforms. Although stretch processing reduces sampling rate, it does so at the expense of increased data collection period. Therefore, the time-bandwidth product of the de-chirped waveform is approximately constant, and stretch processing cannot be considered a compressive technique.**

The sample rate of a stretch processor must match the bandwidth of the beat frequencies created in the de-chirp process. Uniform sampling at lower rates will cause range ambiguities without an anti-aliasing filter, but an anti-aliasing filter removes part of the range swath. We propose a hopping de-chirp signal (sensing matrix) that shifts regions of the range swath in and out of the anti-aliasing filter's bandwidth. This hopping produces a unique time profile of beat frequencies for each range in the swath and enables compressive sampling without sacrificing range swath or resolution and without introducing ambiguities. After describing the hopping de-chirp technique, we analyze multichannel implementations where it is possible to use a different hopping sequence on each channel.

I. Introduction

In traditional radar, linear frequency modulated (LFM) waveforms are often captured with a de-chirp procedure called stretch processing [1]. Stretch processing allows the rate of the analog-to-digital converter (ADC) to be reduced, at the expense of increased data collection time. Because the time-bandwidth product of the received signal is the product of the data collection interval with the Nyquist sampling rate, the trade-off provided by stretch processing cannot be considered a compressive sampling implementation. Indeed, in order to realize a reduced sampling rate via stretch processing, the radar pulse width must be longer than the propagation time across the desired range swath, so the total number of data samples remains approximately the same.

In this paper, we consider the de-chirp process as part of the sensing matrix formulation for compressive sensing (CS), but modify the de-chirp reference signal in a way that enables compressive sampling [2]. Typically, the sample rate must be matched to the bandwidth of beat frequencies produced by the de-chirp process. If the de-chirped signal is sampled at a lower rate, part of the range swath must either be filtered out by an anti-aliasing filter or range ambiguities will be produced. To avoid this, we propose a hopping de-chirp signal that sequentially shifts different regions of the range swath in and out of the anti-aliasing filter's bandwidth (i.e., the de-chirp signal's reference range changes with time). This approach produces a unique time profile of beat frequencies for every range in the desired swath, thus enabling compressive sampling with the possibility of incorporating prior knowledge, without sacrificing range swath, degrading range resolution, or introducing ambiguities. After describing the hopping de-chirp receiver, we consider a multichannel version where each channel can potentially have a different hopping sequence.

In Section II we introduce the compressively sampled stretch processing receiver and show through the use of a hopping de-chirp reference signal that we can compressively sample a LFM pulse. We examine the effects of random hopping and discuss an approach for maximizing the expected energy from the target return through prior knowledge of the target's range distribution. We extend the receiver architecture to a multichannel system and show range/angle ambiguity functions for the random compressive sampling architecture. In Section III we show expected SNR results from the maximization process described in Section II. We make our conclusions in Section IV.

II. Compressively Sampled Stretch Processing Receiver

The modulated pulse is a common radar signal that provides high bandwidth in a long pulse with plenty of energy for detection. One popular type of modulation is linear frequency modulation (LFM)

$$x(t) = \mathrm{rect}\left(\frac{t-0.5T_p}{T_p}\right)e^{j\left(2\pi F_0 t+\pi\gamma t^2\right)},$$

so called because the instantaneous frequency $(F_0 + \gamma t)$ changes linearly with time. The slope, γ called the chirp rate, is given by $\gamma = \frac{B}{T_p}$ where B is the pulse bandwidth and T_p is the pulsewidth.

A. Stretch Processing

The received pulse is shifted by a time τ proportional to target range, so that the signal becomes

$$r(t;\tau) = \alpha\mathrm{rect}\left(\frac{t-0.5T_p-\tau}{T_p}\right)e^{j\pi\gamma(t-\tau)^2}e^{-j2\pi F_0\tau}. \quad (1)$$

The required ADC sample rate can be reduced by "de-chirping" the signal prior to sampling. This is done by multiplying the received signal by the conjugate of a reference signal. The reference signal $x_0(t)$ is offset by τ_0, the center delay of the range swath under interrogation

$$x_0(t) = e^{j\pi\gamma(t-\tau_0)^2}.$$

Multiplying the received signal by the conjugate reference ($x_0^*(t)r(t;\tau) \propto e^{j2\pi\gamma(\tau_0-\tau)t}$) produces a beat frequency ($\gamma(\tau_0-\tau)$) that is proportional to range, relative to the reference [1]. The resulting bandwidth (called the IF bandwidth) generated by all ranges in the swath after de-chirp is

$$B_{IF} = \gamma T_s = B(T_s/T_p) \qquad (2)$$

where $T_s = \tau_{max} - \tau_{min}$ is time width of the range swath.

B. Compressive Sampling

Equation (2) shows that as long as T_s is smaller than the pulsewidth, the Nyquist sample bandwidth (B_{IF}) will be less than the bandwidth of the transmit pulse. If we wish to sample at a lower rate or interrogate a larger swath but keep the same sample rate, we must find a way to modulate the highest beat frequencies into the sample bandwidth. This can be done by "hopping" the reference signal. The beat frequencies of the de-chirped signal will hop with the reference so that different ranges will have unique hopping patterns, thus eliminating range aliasing while ensuring at least part of the return pulse will shift into the sample bandwidth for any given range. The price paid for this reduced-rate implementation is that not all ranges can be shifted within the sample bandwidth at all times; therefore, some energy will be lost when the IF signal passes through the anti-aliasing filter prior to sampling.

Let a hopping de-chirp reference signal be given by

$$x_0(t) = e^{j\pi\gamma(t+\beta(t)-\tau_0)^2}$$

where $\beta(t)$ represents a piecewise hop function that is constant for the time interval $t_{n-1} \le t < t_n$. At the end of each time interval, $\beta(t)$ hops to a new constant value. The t_n's form a set of disjoint time intervals that cover the full sampling period of duration ($T_s + T_p$). The set of $\beta_n = \beta(t_n)$ can either be randomly generated or deterministically designed in order to capture some attribute of the received signal.

In (1) we model the received signal as a finite-energy, bandlimited continuous-time pulse, parameterized by time-of-arrival τ and amplitude α so that $r(t;\tau) = \alpha x(t-\tau)$. In order to quantify the average SNR loss incurred by our compressive implementation (Note that all compressive RF receivers will incur SNR loss), let τ be a random variable with pdf $p_\tau(\tau)$. The received signal $\alpha x(t-\tau)$ is now a random process and the output of the de-chirping process and anti-aliasing filter is

$$y(t;\tau) = h(t) * \left(x_0^*(t) \cdot \alpha x(t-\tau)\right). \qquad (3)$$

where $h(t)$ represents an ideal low pass filter with cutoff frequency matched to half the ADC rate. Defining $E_s = |\alpha|^2$, the expected energy in the received signal is

$$\bar{E}_s = \mathbb{E}_{(\tau)}\left[\int_{\tau}^{\tau+T_p} |y(t;\tau)|^2 \; dt\right]. \qquad (4)$$

In other words, the expected signal energy for a given hopping sequence is calculated by computing the energy retained by the hopping sequence for a given range, and then average over the pdf of the target range. Define the time-bandwidth product (TBWP) of the received signal over T_s as $K = \lfloor T_s B \rfloor$. Divide the range swath into a bank of K range bins defined by $\tau_i = i\frac{T_s}{K} + \tau_{min}$ for $i = 0, \ldots, K-1$. The probability that a target falls in a particular range bin is $p_i = \int_{\tau_i}^{\tau_{i+1}} p_\tau(\tau)d\tau$. Likewise, define the TBWP of the radar pulse as $P = \lfloor T_p B \rfloor$ and divide the received signal into $m = 0, \ldots, P-1$ hop intervals. Here, the de-chirp waveform stays at a particular hop for an interval of $\frac{T_p}{P}$. A shorter hop interval risks unintentionally modulating the signal such that it is no longer identifiable as hopping LFM. Equation (4) becomes

$$\bar{E}_s = E_s \sum_{i=0}^{K-1} p_i \sum_{m=0}^{P-1} \int_{\tau_i+\frac{mT_p}{P}}^{\tau_i+\frac{(m+1)T_p}{P}} |h(t) * e^{j2\pi\gamma(\beta_{m+i}+\tau_0-\tau_i)t}|^2 \; dt.$$

Using Parseval's Theorem [3], the expression can be written in the frequency domain, but because the time integration has been approximated with discrete intervals, we can only express it as an inequality. By swapping the integration with a summation and substituting n for $m+i$ and summing the intervals over $T_p + T_s$, we can integrate over frequency in each interval independently. Define $Q = \min(K, P)$ as the number range bins present in any one time interval, then

$$\bar{E}_s \le E_s \frac{\gamma T_p}{P} \sum_{n=0}^{P+K-1} \int_{-\infty}^{\infty} \sum_{i=0}^{Q-1} p_{n-i} \; H(f) X_{n,i}(f) df, \qquad (5)$$

where $X_{n,i}(f) = \delta\left(f - \gamma(\beta_n + \tau_0 - \tau_{n-i})\right)$ and $\delta(f)$ is the Dirac delta function over frequency.

Figures 1 and 2 show the process of de-chirping a signal using a uniform randomly hopping de-chirp reference. The left panel shows instantaneous frequency versus time for the hopping de-chirp reference and the reflected signal while the right panels shows the de-chirped beat frequency versus time. As seen in Figure 1, for uniformly random hopping $\gamma\beta_n$, and $T_s > T_p$ the reference range for any hop should be within the range swath. After de-chirp, the beat frequencies for any range will be uniformly distributed over B. The de-chirp operation acts as an addition in frequency of two random variables (the hopping sequence and the random target range), each uniformly distributed across B. Since target range and hopping sequence are statistically independent, the resulting distribution on the beat frequencies is triangular, where the base of the triangle is across twice the original bandwidth. The SNR loss during each hop interval is found by integrating the beat frequency distribution over the sample bandwidth. This results in each range suffering the same average loss in SNR.

For $T_s < T_p$, the story is somewhat different. Observe in Figure 2 that each of the possible returns has a constant frequency offset from the center of B_{IF}. Because of this, the $\gamma\tau_{n-i}$ will have the same distribution as τ except scaled by γ and shifted to the center of B_{IF}. Therefore, an efficient hopping pattern hops uniformly around B_{IF}. The expected

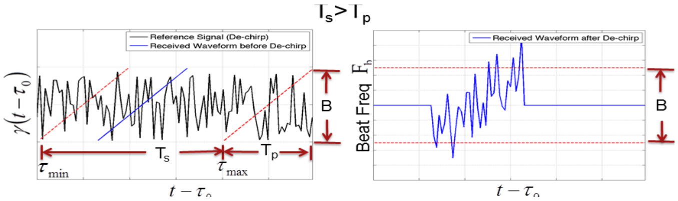

Fig. 1. An example of a uniformly random hopping de-chirp. Reference signal "hops" are shown vs. time, as well as the beat frequency time profile that results from the de-chirping process. Here $T_s > T_p$, so that the beat frequency will vary over B the bandwidth of the received signal

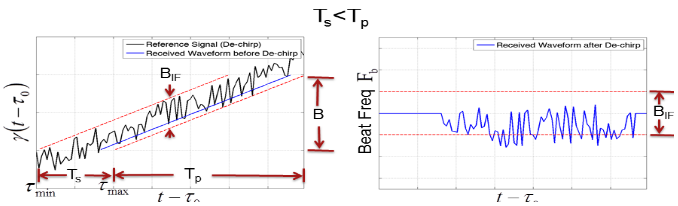

Fig. 2. An example of a uniformly random de-chirp, its frequency "hops" plotted vs. time and the results of de-chirping on a typical receive pulse, its beat frequencies plotted against time. Here $T_s < T_p$, so that de-chirp will vary over B_{IF} the implied bandwidth of range swath at any given moment

loss is again found by finding the distribution of beat frequencies that results from uniformly distributed hops and the range pdf, and then integrating over the sample bandwidth. In this case, however, each range can suffer a different loss in SNR, especially at the endpoints of the swath.

Define the compression ratio C as the ratio of the IF Bandwidth to the compressive receiver's sample rate $C = B_{IF}/F_s$. For random de-chirp, with any range distribution, the loss in SNR is at least C. For uniformly distributed target range, the expected loss in SNR will never be better than C even for custom-designed hopping sequences. For non-uniform distributions it's possible to decrease the expected loss.

Though expressed as an inequality, (5) can be used to maximize the expected energy for a given range distribution and thus the expected SNR. At the n'th interval β_n is chosen so as to maximize the integration in (5) over frequency. This will maximize the expected energy in the sample bandwidth across the range swath, the equivalent of enhancing the most likely ranges.

To express the de-chirping procedure in a matrix form consistent with the CS literature, we form the column vector $\mathbf{y} = [y_0 \,...\, y_N]^T$ from samples of (3) where $y_n = y(t;\tau)|_{t=n/F_s}$, and samples are taken at rate F_s. Thus, the sensing-matrix based representation of (3) becomes

$$\mathbf{y} = \mathbf{K}\mathbf{r}(\tau) = \mathbf{H}\mathbf{X}_0^H\mathbf{r}(\tau), \tag{6}$$

where $\mathbf{H}$ represents filtering followed by sampling at rate F_s. The de-chirp signal $x_0(t)$, either designed or random, is embedded along the diagonal of the matrix-operator $\mathbf{X_0}$, with the off-diagonal elements left as zero. The de-chirp, filtering, and sampling processes can be combined into the measurement matrix-operator $\mathbf{K}$. We model the analog signal $r(t;\tau)$ via samples over the range swath plus the pulsewidth. These samples are generated at a rate sufficiently high (in the Nyquist sense) for the transmit pulse, the hopping de-chirp reference, and the de-chirped signal. Note that the de-chirp output can have bandwidth greater than the original pulse due to the hopping reference (consistent with the tails of the beat frequency pdf described earlier). The received signal becomes

$$\mathbf{r}(\tau) = [r(t_n;\tau)]^T_{n=0,\ldots,D-1}. \tag{7}$$

The simulated de-chirp reference (sampled over the same rate and period) is placed along the diagonal of the square matrix $\mathbf{X_0}$, thus acting as a mapping between spaces of equal size.

In conventional stretch processing, the sample space of $\mathbf{y}$ would be of dimension $N = \lfloor (T_p + T_s)\min(B_{IF}, B) \rfloor$. Here, $\mathbf{K}$ acts to map the higher dimensional space into a space of size $M = \lfloor (T_p + T_s)F_s \rfloor$ with $\mathbf{y}$ a vector of size M. The compression ratio is then $C = \frac{N}{M}$. After compression, a matched filter is formed from the hypothesized delay τ_i,

$$\zeta_i = ((\mathbf{K}\mathbf{K}^H)^{-1}\mathbf{s}_i)^H\mathbf{y} \tag{8}$$

where $\mathbf{s}_i = \mathbf{K}\mathbf{r}(\tau_i)$ and i represents one of K filters from (5).

C. Multichannel Compressive Sampling

For a multichannel system with $1 \times L$ array manifold vector $\mathbf{a}(\psi) = [1 \; e^{j\psi} \; \cdots \; e^{j(L-1)\psi}]$ where ψ is the per-element angle-of-arrival dependent phase shift due to a target signal incident on a uniform linear array [4]. Modify (1) to account for $\mathbf{a}$'s phase progression, then $\mathbf{r}(t;\tau,\psi) = \mathbf{a}(\psi)r(t;\tau)$. The discrete-time version is $\mathbf{R}(\tau,\psi) = \mathbf{r}(\tau)\mathbf{a}(\psi)$, and (6) becomes

$$\mathbf{Y} = \mathbf{K}\mathbf{R}(\tau,\psi) = \mathbf{H}\mathbf{X}_0^H\mathbf{R}(\tau,\psi). \tag{9}$$

The channel responses are the columns of $\mathbf{Y}$, a $M \times L$ matrix.

In the case where $\mathbf{K}$ is identical across channels, the single channel data vector from (6) is sufficient to form the multichannel data matrix $\mathbf{Y} = \mathbf{y}\mathbf{a}(\psi)$. However, when each channel has a different kernel $\mathbf{K}_l$, we must separate the compression effects. The pre-dechirping signal on each channel is given by

$$\mathbf{r}_l(\tau,\psi) = \mathbf{r}(\tau)a_l(\psi) = \mathbf{r}(\tau)e^{j(l-1)\psi}.$$

The temporal measurements on the lth channel are

$$\mathbf{y}_l = \mathbf{K}_l\mathbf{r}_l(\tau,\psi) = \mathbf{H}\mathbf{X}_{0,l}^H\mathbf{r}_l(\tau,\psi)$$

where $\mathbf{X}_{0,l}^H$ is the lth channel de-chirp reference signal matrix, and $\mathbf{Y} = [\mathbf{y}_1 \ldots \mathbf{y}_L]$ is the $M \times L$ data matrix.

The effects of uniformly random independent sampling kernels (i.e., independent hopping sequences on each channel) at $C = 4$ are seen in Figure 3. The ambiguity function $A(\tau,\Psi) = \left|\mathsf{vec}(\mathbf{Y}(\tau=\tau_0,\psi=0))^H\mathsf{vec}(\mathbf{Y}(\tau,\psi))\right|$ is shown for a six channel system, along with cuts through $\tau = \tau_0$ and $\psi = 0$. The mainlobe response is centered at $(\tau_0, 0)$ as expected; however, distortion is visible in the sidelobes due to the random peaks and nulls of the independent kernels. Although the mainlobe of this range-angle ambiguity function is well formed, random high sidelobes could leak across the filter bank causing uncompensated range-dependent distortion. In other words, Figure 3 shows that using different compression kernels on each channel causes the range-angle ambiguity function to be no longer separable in range and angle.

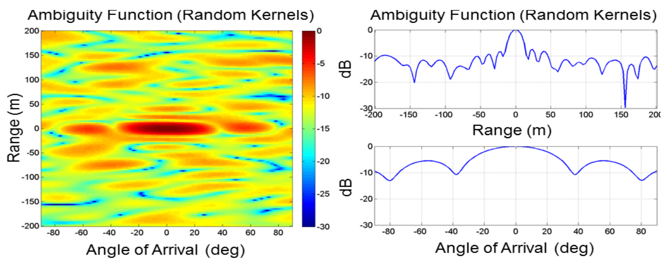

Fig. 3. A plot of the ambiguity surface of $\mathbf{Y}$ centered on τ_0 is shown on the left with cuts through range ($\psi = 0$) and angle ($\tau = \tau_0$) shown to the right.

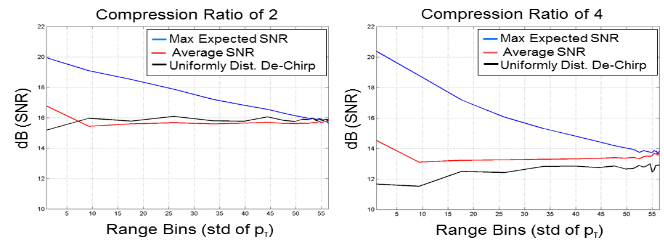

Fig. 4. A plot of SNR vs. the standard deviation of a Gaussian distributed target range measured in range bins for both a 2:1 and 4:1 compression (one range bin = 15m). Shown are the maximum expected SNR from the designed de-chirp plus the average SNR and SNR given a uniformly random de-chirp for a signal that would be 20dB SNR with no compression.

III. Simulation Results

We use a Bayesian signal model, where the time-of-arrival τ_i is a random variable governed by a known distribution

$$\mathbf{y} = \mathbf{K}\left(\mathbf{r}(\tau_i) + \mathbf{n}\right).$$

The noise vector $\mathbf{n}$ is sampled as in (7) from a complex Gaussian random process distributed as $n(t) \sim CN(0, \sigma_n^2)$.

SNR can be measured by examining the peak value of the noise free measurement to the noise increase through the system. For the single channel system, the variance is given by $\sigma_n^2 = \mathbb{E}\left[n^*(t)n(t)\right]$, the expected SNR is

$$\mathbb{E}[\text{SNR}] = \frac{\mathbb{E}_{(\tau_i)}[\zeta_i^* \zeta_i]}{\mathbb{E}[\zeta_n^* \zeta_n]} \tag{10}$$

where $\zeta_n = ((\mathbf{K}\mathbf{K}^H)^{-1} s_i)^H \mathbf{K} n$. For uniformly distributed delays this results in a best case of $\mathbb{E}[\text{SNR}] = \frac{1}{C}\frac{E_s}{\sigma_n^2}$. As the delay becomes less uncertain, we expect the SNR to increase to its uncompressed level $\frac{E_s}{\sigma_n^2}$.

Figure 4 shows the effects on SNR for the $T_s > T_p$ case. The simulation was run with a 10Mhz LFM waveform with 20dB amplitude and $\sigma_n^2 = 1$. The compression is set to $C = 2$ in the left panel and $C = 4$ in the right. The range swath is twice the pulsewidth, $20\mu s$ and $10\mu s$ respectively. Two different types of de-chirp reference signals were used, one from the process described in II-B for expected energy maximization and the other, a uniformly randomly distributed de-chirp. With no compression (Nyquist rate sampling) we expect a 20dB SNR for both average and expected SNR (losses are with respect to this value).

The delay (τ) is distributed as Gaussian with mean τ_0 and standard deviation varying along the x-axis and scaled to range bins. For each point along the x-axis we form a new reference for both types of de-chirps, designed and random. A target is placed separately in each range bin in the swath, and the SNR is computed for both de-chirps at that range-bin.

The average SNR (the red line in both graphs) is found by taking the average SNR across the swath using the designed de-chirp. We expect a loss of 3dB in the left panel and 6dB in the right (based on the compression ratio). However, it is slightly higher (1-2dB) in both graphs due to modulation effects creating by "hopping" over very short time intervals (many successive hops at short time intervals will cause signal energy to leak out of the sample bandwidth).

We use (10) and the distribution of τ to compute the expected SNR (the blue line) for each point on the x-axis. The expected SNR increases with decreasing uncertainty about the target location, until it reaches it's maximum value of 20dB. Thus showing that a hopping de-chirp measurement kernel can be designed to exploit prior information (the distribution of τ) about target locations. For example, the hopping sequence might be optimized to emphasize roads and de-emphasize shadowed ranges.

The expected SNR from a uniformly random de-chirp (the black line) is also shown. Here we predicted a loss of 4 dB in the left panel and 6.3 dB in the right (based on the expected distribution after de-chirp). The slightly higher losses than expected in the right hand graph are again from modulation (high hopping rates).

IV. Conclusion

We have evaluated the effects of random as well as structured hopping for a possible implementation of compressing sampling stretched processing. We have explored the possibility of extending this concept to a multi-channel architecture, and shown that while independent kernels can be compensated in mainlobe, in the sidelobe distortion as a function of range will remain. We have shown that the process described in II-B is capable of increasing expected SNR based on prior information; however, the effects from switching at high rates have yet to be incorporated. It is expected that degradation in SNR should occur if de-chirps are formed that switch at every possible time interval. Further research will incorporate modulation effects as well as adding constraints that will minimize the number of switches within the algorithm.

Acknowledgment

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