Analysis of Weather Radar Return

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Abstract—A mathematical model of detected clutter from an airborne weather radar of conventional design is developed. The model is the joint probability density of samples of radar return from hydrometeors at the same nominal range and scan angle. It is developed from analysis of the effect on the received signal of the following parameters: inhomogeneous hydrometeor motion, radar frequency stability, pulsewidth, antenna beamwidth, scan angle, scan rate, and aircraft speed. In addition, the influence of finite pulse volume on radar sensitivity to hydrometeor motion is examined.

INTRODUCTION

An airborne weather radar is designed to display to the pilot of an aircraft the magnitude of backscatter signal received from hydrometeors as a function of their range and azimuth angle. Since the radar cross section of hydrometeors is proportional to their size and density and since the hydrometeors tend to fall to the earth under the influence of gravity, the display can be interpreted as relative precipitation rate as a function of range and azimuth angle. Rain and hail, the hydrometeors with greatest radar cross section, are frequently responsible for the significant radar backscatter received and displayed by an airborne weather radar.

Guides to the use of airborne weather radars usually state that strong turbulence is associated with high rainfall rates and with high gradients of rainfall rate. They imply that turbulence may be avoided if these areas are circumnavigated. However, the experimental correlation of turbulence with rainfall rate or gradient of rainfall rate is not consistent. Good correlation between turbulence and rainfall rate (but not its gradient) is claimed for two experimental programs [1, 2, p. 17 during which thunderstorms in the mature and decaying stages were penetrated by instrumented aircraft. However, there are radar meteorologists [3] who deny the validity of such conclusions. They state that the claimed correlations of turbulence with rainfall rate or the gradient of rainfall rate “are merely qualitative associations which fail under a variety of physical conditions.”

Turbulence considerations are of considerable importance in aircraft flight planning. At minimum, turbulence may lead to passenger discomfort while, at worst, turbulence may imperil the structural integrity of the aircraft. In the foreseeable future the airborne weather radar will remain the principal means for inflight evaluation of the potential for turbulence along the air route. At present, the potential for turbulence is inferred from the plan position display of received signal strength. There is no obvious barrier to a less circumstantial evaluation of air turbulence by means of a weather radar.

Radar backscatter from hydrometeors in an inhomogeneous turbulent wind field is analyzed, and a model of clutter at the detector output terminal is developed. The model is the joint probability density of two samples of weather radar return from hydrometeors at the same nominal range and scan angle. The model is developed in order to better understand the dependence of clutter fluctuations on turbulence and on other variables. These other variables are subject to partial control by the radar designer or by the pilot. They include antenna beamwidth, scan angle, and scan rate; transmitter pulsewidth and frequency stability; and aircraft speed.

The effect on clutter fluctuations of many of these variables, including turbulence, has been examined previously. The earliest, readily available analyses of radar clutter appear in volumes 13 [4, pp. 553–562] and 24 [5, pp. 124–130] of the M.I.T. Radiation Laboratory Series. The analyses presented in these volumes are frequently referred to in later works on the subject, e.g., [6, pp. 205–213], [7], [8]. However, the analysis in the volume edited by Kerr must exclude from consideration the effect of an inhomogeneous wind field on rain clutter since the amplitudes of orthogonal phasors into which the received signal is resolved are assumed uncorrelated. The analysis of radar clutter in the volume edited by Lawson and Uhlenbeck, although more general, is unfortunately too brief to be of much use to the radar engineer. With the possible exception of inhomogeneous turbulence manifested by a skewed velocity distribution, most variables influencing clutter fluctuations have been studied individually through their contributions to a correlation function or a correlation coefficient. However, since the provision for inhomogeneous turbulence pervades this analysis, the influence of all variables on the clutter model is reexamined rather than assembled from the works of others.

The clutter model is also developed as an aid in the evaluation of techniques for estimating air turbulence from clutter fluctuations. Since the most accurate and theoretically significant measurements of atmospheric turbulence are made with instrumented aircraft, the influence of finite pulse volume on the variance of velocities of hydrometeors from which return is received is examined.

The clutter model is derived by analyzing the sum of signals backscattered from statistically independent hydrometeors. For convenience of analysis and in order that the magnitude of signal from a hydrometeor be reasonably independent of its rate of change of phase, a function of position in the antenna beam, it is assumed that hydrometeors are uniformly distributed throughout an illuminated volume bounded by the leading and trailing edges of the transmitted pulse and by an ideal antenna pattern. It is shown that the signal into the mixer is the sum of orthogonal phasors whose amplitude is a function of expected signal phase which, in turn, is a function of turbulence, of radar design, and of aircraft speed. The joint probability density of pairs of clutter samples is derived from the corresponding density function of phasor amplitudes.
at the mixer input by transforming the signal through the IF amplifier and then through a linear envelope detector.

ANALYSIS

RADAR RETURN FROM HYDROMETEORS

The airborne weather radar is a noncoherent pulsed radar which is designed to sense, by the magnitude of signal at the detector output, the backscattering radar cross section per unit volume of hydrometeors (usually rain) in a plane sector in front of the aircraft and parallel to the earth, and to display this as a function of range and azimuth angle relative to the aircraft. The radar consists of a pulse modulator and transmitter which are periodically connected to the antenna through the duplexer. Any backscattered signal received by the antenna is routed to the mixer through the duplexer and then to an IF amplifier followed by a linear noncoherent envelope detector. The gain of the IF amplifier is periodically increased by sensitivity time control (STC) after each pulse transmitted so that radar return, if from uniformly distributed hydrometeors, is independent of range at the output of the IF amplifier and following detector. Since the propagation velocity of electromagnetic radiation is a finite constant and since the antenna pattern is a pencil beam, the magnitude of signal at the detector output can easily be correlated with the range and azimuth angle of a corresponding illuminated volume by measuring the time elapsed since the last pulse transmitted and the output of an azimuth scan angle sensor. The weather radar antenna is periodically scanned through an azimuth angle of $+30^\circ$ or $-60^\circ$. The scan plane is usually horizontal because of straight and level aircraft flight or because the antenna is stabilized. A block diagram of the assumed weather radar is shown in Fig. 1. Processing for display follows the detector. Display processing is not important to the analysis and will not be discussed.

The detector output is the envelope of the sum of signals received at any time from a large number of hydrometeors which were illuminated a short time earlier. Backscatter received at time $t'$ after the leading edge of the transmitted pulse is from hydrometeors for which round trip propagation time is between $r'$ and $r_p - t_p$, where $r_p$ is transmitted pulse-width. The illuminated volume from which backscatter is received is bounded by spherical shells which define the minimum and maximum ranges to hydrometeors. Their radii differ by $w = c t_p/2$, where $c$ is the velocity of propagation of electromagnetic radiation. The illuminated volume is also bounded by the idealized antenna pattern as shown in Fig. 2(a). In the coordinate system shown in this figure and in Fig. 2(b), the $x$ axis is along the longitudinal axis of the aircraft and is positive in the direction of motion. The $z$ axis is vertical to the earth's surface, positive upward. The $y$ axis is such that the system is right-handed. As shown in Fig. 2(a), the pencil antenna beam is a right circular cone symmetrical about axis $x_1$. The 3 dB width of the beam is $\beta$. The antenna is scanned periodically in the $xy$ plane. The vector position $r$ of any hydrometeor from which backscatter is received at time $t'$ after the leading edge of the last pulse transmitted is, for small $\theta$,

\[
r = r' = r \hat{r} = r [\cos \psi - \theta \sin \psi \cos \phi] + \hat{j} (\sin \psi + \theta \cos \psi \cos \phi) + \hat{z} \theta \sin \phi \]

\[
c(t' - t_p)/2 \leq r \leq ct'/2 \]

\[
0 \leq \theta < \beta/2
\]

\[
0 \leq \phi < 2\pi
\]

where $r$, $\theta$, and $\phi$ are spherical coordinates of a hydrometeor referenced to the antenna, and $\psi$ is the antenna scan angle as shown in Fig. 2(b). The circumflex ($\hat{\cdot}$) signifies a "unit" vector.

The signal $U(t)$ received at the mixer input is the sum of signals backscattered from all hydrometeors within the
Gaussian function of analysis. The antenna pattern is more realistically represented by a illuminated volume. Therefore, since independent and are uniformly distributed throughout the subscripts are statistically independent. Since is influenced by gravitational force as well as by local drafts. Horizontal components of velocity of hydrometeors are those corresponds to a fall velocity \[9\] of 5 to 7 m/s. Experimentally, \[27\] are defined as follows:

\[
U(t) = \sum_k e_k \cos \left( \frac{\omega_n t}{c} + \int_0^t \frac{dr_k}{dt} dt - \frac{2\omega_n r_k(0)}{c} + \alpha_k + e_n \right)
\]

(5)

where, because return from one hydrometeor after several consecutive pulses is of interest, \(t\) is defined to be the time from the leading edge of the first of \(n\) consecutive pulses to illuminate a hydrometeor in the summation. If \(t'\) is time after the last pulse transmitted, as defined for (2), then

\[
t = t' + (n - 1)t_r
\]

(6)

where \(t_r\) is the pulse repetition period. Other variables in (5) are defined as follows:

- \(\omega_n\) is the angular frequency of the transmitted carrier of the \(n\)th pulse transmitted (\(\omega_0\) is the expected value of \(\omega_n\)).
- \(r_k, r_k(0)\) is the range to the \(k\)th hydrometeor at time \(t\), at \(t = 0\).
- \(\alpha_k\) is the phase change of signal on reflection from the \(k\)th hydrometeor.
- \(e_n'\) is the random phase term which models a non-coherent radar. It is uniformly distributed over \(2\pi\) rad and is statistically independent of \(e_m'\) if \(n \neq m\). Subscripts \(n\) and \(m\) are the number of pulses since \(t = 0\) in the set of consecutive radar returns from a hydrometeor.
- \(e_k\) is the magnitude of signal backscattered from the \(k\)th hydrometeor. Because of the idealized antenna pattern and except for minor changes in range attenuation which will be neglected, \(e_k\) is not a function of \(r, \theta,\) and \(\phi\) i.e., of location within the illuminated volume.\(^1\)

It is assumed in this paper that hydrometeors are statistically independent and are uniformly distributed throughout the illuminated volume. Therefore, since \(\omega_0 t_p / 2 \gg 2\pi, 2\omega_n r_k(0)/c\) is asymptotically uniform over \(2\pi\) rad and terms with different subscripts are statistically independent. Since \(e_k\) is independent of position in the illuminated volume, it is independent of \(2\omega_n r_k(0)/c\). \(e_k\) is almost independent of \(dr_k/dt\) in (5). It is assumed that, except possibly for large hailstones, the horizontal components of velocity of hydrometeors are those of the local wind field. The vertical component of velocity is influenced by gravitational force as well as by local drafts. However, in light winds, the Doppler shift of radar return from falling raindrops having the largest backscattering cross section corresponds to a fall velocity \[9\] of 5 to 7 m/s. Experimentally, there is negligible radar return in a light wind from raindrops whose fall velocity is more than 9 m/s or less than 1 m/s.

Updraft or downdraft decreases or increases hydrometeor fall velocity by translation of local reference. The range of vertical velocities is small and, in an airborne weather radar whose antenna beam is horizontal, the radial component of vertical velocity is even smaller. The radial component of velocity \(dr_k/dt\) of a hydrometeor is almost independent of its size. Therefore, the magnitude of signal from a hydrometeor is almost independent of its rate of change of phase in (5).

Define Doppler frequency \(\omega_d\) as the average frequency of \(U(t)\) in (5) less that transmitted, or equivalently, as the expected rate of change of phase of return from the \(k\)th hydrometeor minus \(\omega_n\). Therefore,

\[
\omega_d = \frac{2\omega_n}{c} E\left[ \frac{dr_k}{dt} \right] = -\frac{2\omega_n}{c} E[V \cdot \hat{r}_k]
\]

(7)

where \(E[V \cdot \hat{r}_k]\) signifies the mathematical expectation of scalar product \(V \cdot \hat{r}_k\) over the range of variables \(r, \theta,\) and \(\phi\) defined in (2), (3), and (4), where \(e_k\) is nonzero. To a first approximation, \(dr_k/dt = V \cdot \hat{r}_k\). Both \(V,\) the hydrometeor velocity field, and \(\hat{r}_k,\) the unit position vector of the \(k\)th hydrometeor, are functions of \(r, \theta, \) and \(\phi\).

Let the difference in range at \(t = 0\) between the most distant hydrometeor illuminated and the \(k\)th hydrometeor be

\[
\Delta r_k(0) = \frac{ct'}{2} - r_k(0).
\]

(8)

Also, let

\[
e_n = e_n' - \omega_n \Delta r_k(0).
\]

(9)

Substituting equations (7), (8), and (9) into (5)

\[
U(t) = \sum_k e_k \cos \left( \frac{\omega_n + \omega_d}{2} \right) t
- \frac{2\omega_n}{c} \int_0^t \left( V \cdot \hat{r}_k - E[V \cdot \hat{r}_k] \right) dt
+ \frac{2\omega_n}{c} \frac{\Delta r_k(0)}{c} + \alpha_k + e_n
\]

(10)

Mixer input signal \(U(t)\) in (10) is the sum of two orthogonal phasors with the same carrier frequency. The random phase term \(e_n\) may be included in the carrier or in the phasor amplitude. The latter procedure will be followed as this will facilitate experimental verification of the model. Also, the significant difference between coherent and noncoherent radar systems can more easily be demonstrated. Let

\[
u_1(t) = \sum_k e_k \cos \left[ \frac{2\omega_n}{c} \int_0^t (V \cdot \hat{r}_k - E[V \cdot \hat{r}_k]) dt \right]
- \frac{2\omega_n \Delta r_k(0)}{c} - \alpha_k
\]

(11)

\(^1\) This is not strictly true of real systems, but is a convenience in analysis. The antenna pattern is more realistically represented by a Gaussian function of \(\theta\) and \(\phi,\) and \(e_k\) varies accordingly.
\[ u_2(t) = \sum_k e_k \sin \left( \frac{2\omega_n}{c} \int_0^t (V \cdot \hat{r}_h - E(V \cdot \hat{r}_h)) \, dt \right) \]
\[ - \frac{2\omega_n \Delta r_h(0)}{c} - \alpha_k \]  

(12)

Then

\[ U(t) = [u_1(t) \cos \varepsilon_n + u_2(t) \sin \varepsilon_n] \cos (\omega_n + \omega_d)t \]
\[ + [-u_1(t) \sin \varepsilon_n + u_2(t) \cos \varepsilon_n] \sin (\omega_n + \omega_d)t. \]  

(13)

\[ U(t) \] in (13) is the sum of two orthogonal phasors whose amplitudes are the terms in brackets. If the implied phase reference in (13) were rotated back through \( \varepsilon_n \) radians, the phasor amplitudes would be \( u_1(t) \) and \( u_2(t) \) given by (11) and (12). \( u_1(t) \) and \( u_2(t) \) are the sum of a large number of independent random variables, and none of these dominate the sum. Therefore, by the central limit theorem, \( u_1(t) \) and \( u_2(t) \) are asymptotically normal [10] with zero mean

\[ E[u_1(t)] = E[u_2(t)] = 0 \]  

(14)

and equal variance

\[ E[u_1^2(t)] = E[u_2^2(t)] = \frac{1}{2} \sum_k E[e_k^2]. \]  

(15)

The desired information about hydrometeor motion and about the wind field responsible for that motion is contained in random variables \( u_1(t) \) and \( u_2(t) \). Since the radar is pulsed, wind field data are sampled. Information about the wind field must be extracted from a small number of these data samples. All recoverable information about the statistical properties of the wind field is contained in the joint probability density function of samples of random phasor amplitudes. These are linear combinations of normal random variables \( u_1(t) \) and \( u_2(t) \). The analysis of weather radar clutter is completed with development of the second probability density function of samples of detector output amplitude. This probability density function can be easily written if the covariances of all variables are known [11, p. 20]. The covariances are linear combinations of the correlation functions of \( u_1(t) \) and \( u_2(t) \). It is observed that \( u_1(t) \) and \( u_2(t) \), defined by (11) and (12), are non-stationary processes. They are functions of range and scan angle. In the evaluation of correlation functions, range and scan angle may be treated as a variable or as a parameter.

Two different correlation functions of the same pairs of random variables are used in the analysis of the detector output signal. The first corresponds to returns received after each pulse transmitted, from minimum range to maximum range, such as might be used for A-scope display. Here, range is a variable and scan angle is a parameter. Correlation functions of \( u_1(t_1) \) and \( u_2(t_2) \) in which range is a variable can be used to evaluate required IF amplifier bandwidth or the influence of IF amplifier distortion and bandwidth on the detector output signal. It will be assumed that the envelope of mixer input is not restricted or distorted by the IF amplifier.

In the second set of correlation functions used in model development, both range and scan angle are parameters. The correlation function is the expected value of products of pairs of variables \( u_1(t_1) \) and \( u_j(t_2) \), at substantially the same range and scan angle, which might normally enter the calculation of the intensity of a display element having corresponding range and scan angle coordinates. The correlation coefficient of the second set of correlation functions is almost entirely a function of the relative hydrometeor velocity field, while the correlation coefficient of the first set of correlation functions is almost entirely a function of the propagation velocity of electromagnetic radiation.

The correlation functions requiring evaluation are expected values of products of

\[ u_i(t_1) = u_{i1} \quad \text{and} \quad u_j(t_2) = u_{j2}, \quad i, j = 1, 2. \]  

(16)

The correlation functions can be partially evaluated without specifying range as a variable or parameter. The near independence of the amplitude and phase or return from a hydrometeor, discussed above, is employed to advantage in these calculations. For example, consider

\[ E[u_{11}u_{12}] = E \left\{ \sum_j e_j \cos \left( \frac{2\omega_m}{c} \int_0^{t_1} (V \cdot \hat{r}_j - E(V \cdot \hat{r}_j)) \, dt \right) \right\} \]
\[ - \frac{2\omega_m \Delta r_j(0)}{c} - \alpha_j \right] \sum_j e_j \cos \left( \frac{2\omega_n}{c} \int_0^{t_2} (V \cdot \hat{r}_j) \right) - E(V \cdot \hat{r}_j) \right] \int_1^{t_2} \left( V \cdot \hat{r}_h \right) \, dt - \frac{2(\omega_n - \omega_m) \Delta r_h}{c} \]  

(17)

where the expected value of products of return from different hydrometeors \((i \neq j)\) is zero because phase terms \( 2\omega_m \Delta r_j(0) \) and \( 2\omega_n \Delta r_j(0) \) are statistically independent and are asymptotically uniform over \( 2\pi \) radians. The transformation

\[ \Delta r_h = \int_0^{t_1} (V \cdot \hat{r}_h - E(V \cdot \hat{r}_h)) \, dt - \Delta r_h(0) \]  

(18)

was used to obtain the last equality in (17). Also, since hydrometeors are independent of one another and since amplitude is independent of phase, the order of expectation and summation may be interchanged, as was done in (17), and the resulting equation expressed as the sum of products of two expectations. Substitute the equation for \( \Delta r_h \), the term in brackets in (1), into (17) and employ the first mean-value theorem for integrals [12, p. 34] to simplify the integral. Let \( \Theta_m \) and \( \Phi_m \) be "mean
values" of $\theta$ and $\phi$ in the sense of the theorem. Also let

\[ A = \frac{2\omega_n}{c} \int_{t_1}^{t_2} (V \cdot \hat{x} - E[V \cdot \hat{x}]) \, dt \]

\[ B = \frac{2\omega_n}{c} \int_{t_1}^{t_2} (V \cdot \hat{y} - E[V \cdot \hat{y}]) \, dt \]

\[ C = \frac{2\omega_n}{c} \left( -\sin \psi \int_{t_1}^{t_2} V \cdot \hat{x} \, dt + \cos \psi \int_{t_1}^{t_2} V \cdot \hat{y} \, dt \right) \]

\[ D = \frac{2\omega_n}{c} \int_{t_1}^{t_2} V \cdot \hat{z} \, dt \]

\[ G = (C^2 + D^2)^{1/2} \]

\[ \xi = \arctan (D/C) \]

\[ \omega_\delta = \omega_n - \omega_m. \]

Substitute (19) into (17) and employ a standard trigonometric identity to find

\[ E[u_{11}u_{12}] = \frac{1}{2} \sum_n E[e_n^2]E \left\{ \cos \left[ A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi) \right] \cos \left( \frac{2\omega_\delta \Delta r_k}{c} \right) \right. \\
+ \sin \left[ A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi) \right] \\
\left. \cdot \sin \left( \frac{2\omega_\delta \Delta r_k}{c} \right) \right\}. \]

(20)

RADAR BEAMWIDTH AND SCAN ANGLE, AND AIRCRAFT SPEED

The arguments of all sine and cosine terms in (20) are functions of hydrometeor position in the illuminated volume and of transmitted frequency. In order to evaluate expected values of these terms, it is necessary to know or to make plausible assumptions about random variables $\Theta$, $\Phi$, $\Delta r$, $\omega_n$, and $\omega_\delta$. It will be assumed that the joint probability density functions of $\Theta$, $\Phi$, and $\Delta r$ are the same as those of $\theta$, $\phi$, and $\Delta r(0)$, and that these random variables are independent of transmitted frequency. Since the $k$th hydrometeor may occupy any position in the illuminated volume with equal probability, the desired probability density function is

\[ f(\Theta, \Phi, \Delta r) = \frac{4\Theta}{\pi \omega \beta^2}, \quad 0 < \Theta < \frac{\beta}{2}; 0 < \Phi < 2\pi; \]

\[ 0 < \Delta r < \omega \]

(21)

where $\beta$ is the antenna beamwidth, $\omega$ is the width of illuminated volume, and $c r^2 / 2 > \omega = c t_p / 2$.

It will be convenient to complete the evaluation of (20) by first finding the expected values over variables $\Theta$, $\Phi$, and $V$ of sine and cosine functions with argument $[A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi)]$. Observe that variables $\Theta$, $\Phi$, and $V$ appear only in this argument and that variables $\Theta$ and $\Phi$ occur only in the third term of the argument. First, consider the expected value of the cosine function. Write the function as the difference of expected values of products of two cosine functions and two sine functions, with one of the arguments being $[A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi)]$. Expand trigonometric functions with this argument in appropriate series of Bessel functions [13, p. 196, formulas 95, 96], and find the expected values over $\Phi$. Next, express the only remaining nonzero term $J_0(\Theta G)$ in a series [13, p. 190, formula 9], and find the expectation over $\Theta$. The result is

\[ E[\cos \left( A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi) \right)] \]

\[ = E \left\{ \cos (A \cos \psi + B \sin \psi) \sum_{n=0}^{\infty} \frac{(-1)^n \beta^{2n} G^{2n}}{(n + 1)4^n n!} \right\}. \]

(22)

Only radar (antenna) parameters have been used thus far to evaluate the correlation function of $u_1(t_1)$ and $u_1(t_2)$. In order to continue the evaluation, it will be necessary to make some assumptions about the hydrometeors, their velocity distribution, and their radar cross section. Radar cross section will be deferred until $E[k_{22}]$ is evaluated, but hydrometeor velocity distribution will be examined next. The expectation in (22) is over the range of components of hydrometeor velocities found in terms $A$, $B$, and $G$ defined in (19). One of the assumptions made in this paper is that the horizontal components of hydrometeor velocity are the horizontal components of the local wind field. The vertical component of hydrometeor velocity differs from that of the local wind field by the terminal fall velocity in stagnant air. As observed earlier, the average Doppler shift of radar return from raindrops in light winds corresponds to a terminal velocity of between 5 and 7 m/s. Updrafts or downdrafts in thunderstorms might reduce or increase this velocity. In any event, the error accepted in the airborne weather radar analysis by equating the vertical component of hydrometeor velocity with that of the wind field, or even by neglecting vertical velocity entirely, is small. $G$, defined in (19), is the phase change resulting from the cross-beam component of hydrometeor velocity. The greatest contribution to $G$ in (22) is made by the aircraft velocity in the integrand $V \cdot \hat{x}$ of the first term in $C$. Unless $\psi = 0$, aircraft velocity dominates the series in (22). Functions $A$ and $B$ in (22), defined in (19), are integrals of horizontal components of the deviation of wind field from the average over the sample space, the illuminated volume. If wind velocity is represented as the sum of a mean velocity and a deviation from mean, it is evident that terms in a series representation of $\cos (A \cos \psi + B \sin \psi)$ are larger than corresponding terms in the series in (22) because the latter are multiplied by powers of $(\beta/4)$, a number much less than
one. From this cursory examination of the relative magnitudes of
velocity terms in (22), it is concluded that terms in the series might reasonably be approximated by average relative (to the aircraft) components of the wind field or by aircraft speed alone, the major component of velocity in the x direction. With this approximation, all deviations of wind field from the mean are confined to the argument of the cosine term. Accordingly, let

$$v_x = V \cdot \hat{x} - E[V \cdot \hat{x}] \quad v_y = V \cdot \hat{y} - E[V \cdot \hat{y}]$$

(23)

$$G = \frac{2\omega_0}{c} V_a(t_2 - t_1) \sin \psi$$

(24)

where \(V_a\) is aircraft speed.

Approximation (24) exposes clutter model features believed characteristic of an airborne weather radar. That is, the clutter correlation function and spectrum are functions of aircraft speed, scan angle, and antenna beamwidth. By replacing the velocity components in function \(G\) by a constant aircraft speed, the indicated integrations are expedited, and the expected value of the series is easily evaluated. The remaining integrals, those in functions \(A\) and \(B\), can be evaluated by taking advantage of Taylor’s hypothesis [14, p. 56]. Except at lower altitudes and large scales, the structure of the wind field persists for some time and is transported by the mean wind. Measurements made at one point as a function of time may be transformed to a function of space at some instant of time. The integrands of \(A\) and \(B\) are substantially constant in time interval \(t_2 - t_1\). Substitute (23) into \(A\) and \(B\) in (22). Obtain

$$\cos \psi \int_{t_1}^{t_2} v_x \, dt + \sin \psi \int_{t_1}^{t_2} v_y \, dt$$

$$= (v_x \cos \psi + v_y \sin \psi)(t_2 - t_1)$$

$$= u(t_2 - t_1)$$

(25)

where \(u\) is the component parallel to the antenna axis of the deviation of wind velocity from the mean. Substitute (24) and (25) into (22), and let \(\tau = t_2 - t_1\) so that

$$E[Cos [A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi)]]$$

$$= E \left[ \cos \left( \frac{2\omega_0 V \tau}{c} \right) \right] \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)(n!)^2} \left( \frac{\beta \omega_0 V_a \tau \sin \psi}{2c} \right) 2^n$$

(26)

**INHOMOGENEOUS TURBULENCE**

The expectation in (26) can be evaluated if the velocity distribution of the wind field is known. Either wind shear or turbulence may give rise to a velocity distribution over the illuminated volume. This is illustrated in Fig. 3 where the x component of velocity is represented. Only turbulence will be considered in this analysis. If the wind is turbulent, the probability density function of random variable \(v\) [14, p. 94] tends to be normal, particularly if turbulence is homogeneous. If turbulence is inhomogeneous, the probability density function is skewed. Most experimental probability density functions of velocity in turbulent flow are based upon laboratory wind tunnel data. Measurements of environmental wind velocity at the higher altitudes and in thunderstorms are difficult and expensive. In a series of measurements [15] of upper atmospheric turbulence, it was found that the velocity probability density function was slightly skewed. \(\gamma_1\), the ratio of third central moment to the cube of standard deviation, a measure of skewness, was as high as 0.22 for one horizontal component of velocity but was 0.04 and 0.05 for the other horizontal component and for the vertical component of velocity. Perhaps of greater significance, \(\gamma_2\), the ratio of the fourth central moment to the square of variance less 3, a measure of excess, was found to be approximately 2 for all components of velocity rather than zero as expected of a normal random variable. That is, the fourth central moment is 60 to 70 percent greater than that of a normal random variable. The significance of \(\gamma_2 > 0\) is not known. A nonzero value for \(\gamma_2\) does make the expectation in (27) more difficult to evaluate and will be neglected. Skewness of the velocity distribution will be considered in this analysis by adding one additional term [16, p. 117] to a normal velocity probability density function. Since horizontal components of the wind field and of hydrometeor velocity are assumed approximately equal, let the probability density function of \(v\), the component parallel to the antenna axis of the deviation from the mean of hydrometeor velocity, be

$$f(v) = \left[ 1 - \frac{\gamma_1}{3!} \left( \frac{3v^3}{\sigma - v^3} \right) \right] \frac{1}{\sqrt{2\pi \sigma}} \exp \left( -\frac{v^2}{2\sigma^2} \right)$$

(27)

where

- \(\sigma^2\) variance of velocity, parallel to the antenna axis, of hydrometeors in the illuminated volume;
- \(\gamma_1\) ratio of the third central moment to the cube of standard deviation.
It is easily seen that the even moments of \( v \) are
\[
E[v^{2m}] = \frac{(2m-1)!}{2^{m-1}(m-1)!} \sigma^{2m}, \quad m = 1, 2, \ldots
\]  
(28)
and the odd moments are
\[
E[v^{2m+1}] = \frac{\gamma_1 (2m+1)!}{3! 2^{m} m!} \sigma^{2m+1}, \quad m = 0, 1, 2, \ldots
\]  
(29)
The expectation in (26) can be evaluated by expanding the cosine term in a series and computing moments with the aid of (28):
\[
E\left[ \cos\left(\frac{2\omega_n \psi}{c}\right) \right] = \exp\left[ -2 \left( \frac{\omega_n \sigma}{c} \right)^2 \right].
\]  
(30)
Since the expected value of the cosine function is a Gaussian function, it will be convenient to approximate the series in (26) by another Gaussian function whose series expansion has the same first and second terms. Make the approximation and substitute (30) into (26). Find
\[
E[\cos \{ A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi) \}]
= E \left[ \exp \left[ -2 \left( \frac{\omega_n \sigma}{c} \right)^2 \right] \right. 
\times \exp \left[ -2 \left( \frac{\omega_n \beta V_a \sigma \sin \psi}{4c} \right)^2 \right] \]  
(31)
Similarly, the expected value of the sine function having the same argument as (31) is evaluated with the aid of (29)
\[
E[\sin \{ A \cos \psi + B \sin \psi + \Theta G \cos (\Phi - \xi) \}]
= E \left[ -\frac{\gamma_1}{3!} \left( \frac{2\omega_n \sigma}{c} \right)^3 \exp \left[ -2 \left( \frac{\delta_n \sigma}{c} \right)^2 \right] \right. 
\times \exp \left[ -2 \left( \frac{\omega_n \beta V_a \sigma \sin \psi}{4c} \right)^2 \right] \]  
(32)
RADAR FREQUENCY STABILITY
The evaluation of expectations (31) and (32) will be completed by finding the expectation over \( \omega_n \). Assume that the carrier frequencies of transmitted microwave pulses are independent, normal random variables with mean and variance:
\[
E[\omega] = \omega_0 \quad \text{and} \quad E[(\omega - \omega_0)^2] = \sigma_\omega^2
\]  
(33)
\( \omega_0 \), defined in the last of equations (19), is also a normal random variable with mean and variance:
\[
E[\omega_0] = 0 \quad \text{and} \quad E[\omega_0^2] = \begin{cases} 
2\sigma_\omega^2, & \text{if } m \neq n \
0, & \text{if } m = n.
\end{cases}
\]  
(34)
Variables \( \omega_n \) and \( \omega_o \) are not independent. Their joint probability density function is
\[
f(\omega_n, \omega_o) = f(\omega_n | \omega_o)f(\omega_o)
= \frac{1}{2\pi \sigma^2} \exp \left[ -\frac{(\omega_n - \omega_0 - \omega_o)^2}{\sigma_\omega^2} \right] \exp \left[ -\frac{\omega_o^2}{4\sigma_\omega^2} \right].
\]  
(35)
Observe that the conditional probability density function \( f(\omega_n | \omega_o) \) has a maximum at \( \omega_n = \omega_0 + \omega_o \), and it is anticipated that the product of (31) or (32) and (35) will be maximum near this value of \( \omega_n \). Therefore, [17, p. 151] find the approximate conditional expected values of (31) and (32) by expanding these functions of \( \omega_n \) in Taylor’s series about \( \omega_n = \omega_0 + \omega_o \), multiply by the conditional (on \( \omega_o \)) probability density function of \( \omega_n \), and integrate. In each case, the series converges rapidly since both \( \omega_0 \) and \( \sigma_\omega \) are of order \( 10^{-5} \omega_0 \). Only the first term of the series need be retained, and this may be simplified with little error by omitting \( \omega_o \). The resulting equations are (31) and (32) with \( \omega_n \) replaced by \( \omega_0 \). Substitute these into (20) and get
\[
E[u_{11} u_{12}] = \frac{1}{2} \sum_k E[e_k^2] E\left[ \cos \left( \frac{2\omega_0 \Delta \tau}{c} \right) \right]
- \frac{\gamma_1}{3!} \left( \frac{2\omega_0 \sigma}{c} \right)^3 \sin \left( \frac{2\omega_o \Delta \tau}{c} \right)
\times \exp \left[ -2 \left( \frac{\omega_0 \sigma^2}{c} \right)^2 \right]
\times \exp \left[ -2 \left( \frac{\omega_0 \beta V_a \sigma^2 \sin \psi}{4c} \right)^2 \right].
\]  
(36)
where the remaining expectations are over \( \omega_o \) and \( \Delta \tau \). Observe that \( \sin (2\omega_o \Delta \tau/c) \) is an odd function of \( \omega_0 \) while both \( \cos (2\omega_o \Delta \tau/c) \) and \( f(\omega_o) \) are even functions of \( \omega_o \). Therefore, the expected value of the sine function is zero, and the expected value of the cosine function is found by methods similar to (30) to be a Gaussian function. The remaining random variable \( \Delta \tau \) is uniformly distributed over \( \omega \) (see (18)). Consequently, the expected value of the Gaussian function can be expressed as an error function with the following results:
\[
E[u_{11} u_{12}] = \frac{1}{2} \sum_k E[e_k^2] \sqrt{\pi} \frac{c}{2 \sigma_\omega w} \exp \left[ -2 \left( \frac{\sqrt{2\sigma_\omega w}}{c} \right)^2 \right]
\times \exp \left[ -2 \left( \frac{\omega_0 \sigma}{c} \right)^2 \right]
\times \exp \left[ -2 \left( \frac{\omega_0 \beta V_a \sigma \sin \psi}{4c} \right)^2 \right].
\]  
(37)
Other correlation functions can be similarly evaluated. It is
readily seen at a stage corresponding to (17) that

\[ E[u_{12} u_{22}] = E[u_{11} u_{22}] . \]  

(38)

The remaining two correlation functions can be evaluated by a development parallel to that leading to (37) and (38), with the following results:

\[
E[u_{11} u_{22}] = -\frac{1}{2} \sum_k E[e_k^2] \sqrt{\frac{\pi}{2}} \frac{c}{2 \sigma_w w} 
\cdot \exp \left[ -2 \left( \frac{\omega_0 \tau}{c} \right)^2 \right] 
\cdot \exp \left[ -2 \left( \Omega w_0 \tau \sin \psi \right)^2 \right] 
\cdot \text{erf} \left( \frac{\sqrt{2} \sigma_w w}{c} \right) \gamma_1 \left( \frac{2 \omega_0 \sigma_T}{c} \right) \right] 
\end{eqnarray}

(39)

and

\[
E[u_{21} u_{12}] = -E[u_{11} u_{22}] . \]  

(40)

Except for \( \Sigma E[e_k^2] \), this completes the evaluation of the correlation functions of normal random variables \( u_1(t) \) and \( u_2(t) \) defined in (11) and (12). The results are summarized in (37)-(40).

**ANTENNA SCAN RATE**

\( \Sigma E[e_k^2] \) is signal power from those hydrometeors common to illuminated volumes from which radar return is received at times \( t_1 \) and \( t_2 \). That is, \( \Sigma E[e_k^2] \) is the sum of signal power contributions from hydrometeors in an intersection volume such as shown in Fig. 4. Hydrometeors outside this volume contribute nothing to the sum of expected values. It was observed after (17) that the expected values of products of return from different hydrometeors is zero. When \( t_2 = t_1 \) or \( \tau = 0 \), the intersection volume is maximum, and \( \Sigma E[e_k^2] \) is the backscattered power from hydrometeors usually evaluated by means of the radar equation [6, pp. 199-205]. If hydrometeors are uniformly distributed in space and ergodicity is assumed, then the time average of the square of \( U(t) \) in (13), the signal power at the mixer input, is

\[
\frac{1}{T} \int_0^T U^2(t) \, dt = \frac{1}{2} \left[ E[u_{11} u_{12}] + E[u_{21} u_{22}] \right] \tau = 0
\]

\[
= \frac{1}{2} \sum_k E[e_k^2] \]  

(41)

where (37)-(40) at \( \tau = 0 \) were used in (41). From the radar equation

\[
\frac{1}{2} \sum_k E[e_k^2] \bigg|_{\tau = 0} = \frac{P_r G^2 \lambda^2}{(4\pi)^3 r^4} \left[ \frac{\pi^2 w_0 \rho^2}{4} \right] n, 
\]

\[ r \gg w, \tau = 0 \]  

(42)

where the first quantity in brackets on the right side of (42) is a function of transmitted power, antenna gain, wavelength, and range in accordance with the radar equation; the second term in brackets is the illuminated volume; and \( \eta \) is the specific reflectivity or radar cross section per unit volume of hydrometeors. Values for \( \eta \) may be found in books on radar design, e.g., [6, p. 203]. Signal power calculated by (42) is too high. A better estimate of received signal power is provided by a formulation of the radar equation which is based on a more realistic representation of the antenna pattern [18, p. 32].

As the intersection volume is reduced because of the velocity of propagation of electromagnetic radiation (Fig. 4(a)) or because of antenna scanning (Fig. 4(b)), \( \Sigma E[e_k^2] \) is reduced in proportion. Random variables \( u_1(t) \) and \( u_2(t) \) are nonstationary, in part, because the illuminated volume (and the intersection volume) is a function of range as seen in (42). As noted earlier, two different correlation functions are distinguished according to whether range \( r \) is a variable or a parameter. In either case, the value at \( \tau = 0 \) is known from (41) and (42). All that remains to be evaluated is the fractional change in intersection volume as \( \tau \rightarrow \infty \). Two volume correlation coefficients are computed. These are the ratio of intersection volume at any elapsed time to the volume at \( \tau = 0 \), the volume illuminated. Both coefficients were approximated by Gaussian functions for mathematical convenience. The method of Galerkin [19, p. 286] was used to find a suitable approximation to the idealized intersection volume. Low order moments of Gaussian function \( \exp(-b\tau^2) \) were matched to those of the intersection volume by choice of \( a \) and \( b \). This was then transformed into a correlation coefficient by dividing by \( a \). Since \( E[e_k^2] \) is nonstationary, the idealized intersection

---

**Fig. 4.** Intersection of illuminated volumes from which return is received at different times. (a) Propagating pulse. (b) Scanning antenna.
volume was evaluated [20], [21] for $t_2 < t_1$, to find the correlation coefficient. However, the significance of $t_2 < t_1$ is lost in the Gaussian functions. The approximate volume correlation coefficient for a propagating pulse is

$$\rho \approx \exp \left( -\frac{27\psi^2 \tau^2}{8g^2} \right). \tag{43}$$

The approximate volume correlation coefficient for a scanning antenna is

$$\rho \approx \exp \left( -2.45 \times 10^{11} \tau^2 \right)$$

$$\rho \approx \exp (-81.0\tau^2). \tag{45}$$

The magnitudes of coefficients of $\tau^2$ in the exponents of Gaussian terms in (37) or (39) are between $10^4$ and $10^7$. The spectral width of radar return from a propagating pulse is almost entirely determined by $\rho_0(\tau, r)$. The spectral width of samples of signal at the same range and same nominal scan angle is almost independent of scan rate but is a function of the wind field, aircraft speed, antenna beamwidth, scan angle, and transmitter frequency stability.

**DETECTED CLUTTER**

The output $X(t)$ of the IF amplifier is the image of the mixer input $U(t)$ in (13) scaled in amplitude and translated in frequency. $U(t)$ is mixed with the local oscillator signal, and the component with difference frequency $\omega_l = \omega_0 - \omega_{LO}$ is amplified, and the other component is attenuated. Except for small nonlinearities from STC, the IF amplifier is a linear device. Since the mixer input is the sum of orthogonal phasors whose amplitudes are the sum of normal random variables, this is also true of the IF output. Let

$$X(t) = \left[ x_1(t) \cos \epsilon_n + x_2(t) \sin \epsilon_n \right] \cos (\omega_1 + \omega_d)t$$

$$+ \left[ -x_1(t) \sin \epsilon_n + x_2(t) \cos \epsilon_n \right] \sin (\omega_1 + \omega_d)t$$

where $\omega_l$ is the intermediate frequency and $\epsilon_n$ is defined by (9). $x_1(t)$ and $x_2(t)$ are components of phasor amplitude corresponding to $u_1(t)$ and $u_2(t)$ in (13). The most significant difference between $x_1(t)$, $x_2(t)$ and $u_1(t)$, $u_2(t)$, is their variance. The variances of $u_1(t)$ and $u_2(t)$ are equal and their sum is given by (42). The variances of $x_1(t)$ and $x_2(t)$ are also equal but, because of STC, are independent of range if hydrometeors are uniformly distributed in range. Let

$$E[x_1^2(t)] = E[x_2^2(t)] = \alpha_x^2 \approx \eta. \tag{46}$$

It is readily seen that the average square of $X(t)$ is $\alpha^2$ just as the average square of $U(t)$ is (42). The voltage scale at amplifier output which should be included in (47) is unknown and is unimportant to this analysis. Also, because the amplifier is linear, $x_1(t)$ and $x_2(t)$ have zero mean:

$$E[x_1(t)] = E[x_2(t)] = 0. \tag{47}$$

If, as expected of a well designed and maintained IF amplifier, there is no distortion, the correlation functions of $x_1(t)$ and $x_2(t)$ at times $t_1$ and $t_2$ can be written from (37)-40), and from (43) or (44). Insufficient amplifier bandwidth reduces output signal power. It also tends to increase the effective transmitted pulsewidth and the volume of wind field whose variance appears in the correlation functions. It is assumed in this analysis that the IF amplifier is free of distortion and has adequate bandwidth. The correlation functions of $x_1(t)$ and $x_2(t)$ are identical to those of $u_1(t)$ and $u_2(t)$ except that variance is replaced by (47). The correlation functions of greatest interest are those of consecutive samples of signal at the same nominal range and scan angle. In order to simply the notation, let

$$\rho(\tau) = \sqrt{\frac{\pi}{2}} \frac{c}{\omega w} \exp \left( -\frac{1}{2} \frac{\omega w}{\omega_0} \right) \cdot \exp \left[ -2 \left( \frac{\omega_0 \sigma_\tau}{c} \right)^2 \right] \cdot \exp \left[ -2 \left( \frac{\omega_0 \beta W \sin \psi}{4c} \right)^2 \right] \cdot \exp \left( -\frac{27\psi^2 \tau^2}{8g^2} \right) \tag{48}$$

$$q(\tau) = \frac{\gamma_1}{3} \left( \frac{\omega w \sigma_\tau}{c} \right)^3 \tag{49}$$

where terms in $\rho(\tau)$ and $q(\tau)$ are taken from (37), (39), and (44). The correlation functions of $x_1(t)$ and $x_2(t)$ are

$$E[x_1(t)x_1(t + \tau)] = E[x_2(t)x_2(t + \tau)] = \alpha_x^2 \rho(\tau). \tag{50}$$

$$E[x_1(t)x_2(t + \tau)] = -E[x_2(t)x_1(t + \tau)] = -\alpha_x^2 q(\tau) \rho(\tau). \tag{51}$$

The joint probability density function of variables $x_1(t_1)$, $x_2(t_2)$, $x_1(t_2)$, $x_2(t_2)$, $\epsilon_i$, and $\epsilon_j$ can be evaluated with the aid
of (48), (51), and (52)
\[
f(x_{11}, x_{21}, x_{12}, x_{22}, e_i, e_j) = \frac{1}{(2\pi)^4 |M|^{1/2}} \exp \left( -\frac{1}{2} [X]^T [M]^{-1} [X] \right)
\]
(53)

\[
f(w_{11}, w_{21}, w_{12}w_{22}, e, e_j)
\]
\[
= \frac{1}{(2\pi)^4 \sigma_x^4 [1 - \rho^2(1 + q^2)]}
\]
\[
\cdot \exp \left( \frac{\left( w_{11}^2 + w_{21}^2 - 2\rho(w_{11}w_{12} \cos e + w_{11}w_{22} \sin e - w_{21}w_{12} \sin e + w_{21}w_{22} \cos e) - 2q\rho(w_{11}w_{12} \sin e - w_{21}w_{12} \cos e + w_{21}w_{22} \sin e) + w_{12}^2 + w_{22}^2 \right)}{2\sigma_x^2 [1 - \rho^2(1 + q^2)]} \right)
\]
(58)

where

[X] column vector whose elements are normal random variables \(x_{11}(t_1), x_{21}(t_1), \ldots\),

[M] covariance matrix whose elements are the covariances of \(x_{nm}(n, m = 1, 2)\) (correlation functions (51) and (52) evaluated at \(\tau = t_2 - t_1\)).

Phase variables \(e_i\) and \(e_j\) are statistically independent and uniformly distributed over \(2\pi\) rad as defined for \(e^i\) in (5). Vector \([X]\) and matrix \([M]\) are

\[
[X] = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} x_1(t_1) \\ x_2(t_1) \\ x_1(t_2) \\ x_2(t_2) \end{bmatrix}
\]

The desired joint probability density is

\[
f(y_1, y_2, \phi_1, \phi_2, e, e_j)
\]
\[
= \frac{y_1 y_2}{(2\pi)^4 \sigma_x^4 [1 - \rho^2(1 + q^2)]} \exp \left( \frac{y_1^2 - 2\rho y_1 y_2 [\cos(\phi_2 - \phi_1 - e) - q \sin(\phi_2 - \phi_1 - e)] + y_2^2}{2\sigma_x^2 [1 - \rho^2(1 + q^2)]} \right)
\]
(60)

\[
[M] = \sigma_x^2 \begin{bmatrix} 1 & 0 & \rho & -\rho q \\ 0 & 1 & \rho q & \rho \\ \rho & \rho q & 1 & 0 \\ -\rho q & \rho & 0 & 1 \end{bmatrix}
\]
(54)

Before integrating over \(\phi_1, \phi_2, e, e_j\), to find the joint probability density of \(y_1\) and \(y_2\), make the transformation

\[
y_1 = y_1^i \quad y_2 = y_2^i
\]
\[
\phi = \phi_2 - \phi_1 - e + \theta 
\]
\[
\phi_1 = \phi_1
\]
\[
e = e_j \quad e_j = e_j
\]
\[
\theta = \arctan q.
\]
(61)

Substitute (54) into (53). Find

\[
f(x_{11}, x_{21}, x_{12}, x_{22}, e_i, e_j)
\]
\[
= \frac{1}{(2\pi)^4 \sigma_x^4 [1 - \rho^2(1 + q^2)]} \exp \left( \frac{-x_{11}^2 - 2\rho x_{11} x_{12} + x_{12}^2 + x_{21}^2 - 2\rho x_{21} x_{22} + x_{22}^2 + 2q\rho x_{11} x_{22} - 2q\rho x_{21} x_{22}}{2\sigma_x^2 [1 - \rho^2(1 + q^2)]} \right)
\]
(55)

The joint probability density function of phasor amplitudes

\[
w_1(t) = x_1(t) \cos e_n + x_2(t) \sin e_n
\]
\[
w_2(t) = -x_1(t) \sin e_n + x_2(t) \cos e_n
\]
The joint probability density function of $y_1$ and $y_2$ is
\[ f(y_1, y_2) = \frac{y_1 y_2}{(2\pi)^4 a_x^4 [1 - \rho^2(1 + q^2)]} \cdot \int_0^{2\pi} d\phi \int_0^{2\pi - \varepsilon} d\varepsilon \cdot \int_0^{2\pi - \phi - \varepsilon} e^{+\theta} \exp \left[ -\frac{y_1^2 + y_2^2}{2a_x^2 [1 - \rho^2(1 + q^2)]} \right] \cdot I_0 \left( \frac{\rho(1 + q^2)^{1/2} y_1 y_2}{a_x^2 [1 - \rho^2(1 + q^2)]} \right). \] (63)

$Y_1$ and $Y_2$ are detector output signals at times $t_1$ and $t_2$. $a_x$ is clutter signal power from the IF amplifier, and $\rho$ and $q$ are defined in (49) and (50). The joint probability density function of detector output signals sampled at different ranges following the same transmitted pulse is also given by (63) with $\rho$ appropriately defined. Let $\sigma_{\omega} = 0$ (see (34)); and, instead of using (44) as the last Gaussian term in (49), use (43). Finally, for $\rho = 0$ it is seen that (63) is the product of two Rayleigh density functions. The output of the noncoherent linear envelope detector is a Rayleigh random variable.

**HYDROMETEOR VARIANCE IN RADAR RETURN**

The analysis of weather radar return is completed with an examination of the significance of $\sigma^2$, the variance of the component parallel to the antenna axis of velocity of hydrometeors in the illuminated volume. The theory of turbulence in fluids is developed in terms of velocity at a point. Also, the most accurate measurements of atmospheric turbulence, made with instrumented aircraft and from meteorological towers, can be considered point measurements since instrument dimensions are less than the smallest scale of turbulence of interest. In order to better understand the functional dependence of weather radar return on turbulence, the power spectral density of theory and experiment is related to the variance of a component of velocity of illuminated hydrometeors. The development parallels that of Rogers and Tripp [22] up to the stage where radar parameters are introduced. It is then shown that the illuminated volume of hydrometeors acts as a high-pass spatial filter on the spectrum of turbulence.

The power spectral density tensor $S_{ij}(\Omega)$ is the Fourier transform of the correlation tensor $R_{ij}(\xi)$ [14, ch. 1], [23, ch. 1]. This correlation tensor is an ordered array of expected values of the products of the velocity components $V_i$ and $V_j$ at points $r$ and $r + \xi$ at the same time. If turbulence is homogeneous, then, by definition,
\[ R_{ij}(\xi) = E[V_i(r) V_j(r + \xi)]. \] (64)

is independent of $r$. The most frequently used transform pair relating the correlation and spectral density tensors is
\[ R_{ij}(\xi) = \int_{-\infty}^{\infty} \exp \left( j\xi \cdot \Omega \right) S_{ij}(\Omega) \, d\Omega \] (65)
\[ S_{ij}(\Omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp \left( -j\xi \cdot \Omega \right) R_{ij}(\xi) \, d\xi. \] (66)

Only one of the tensor components of the correlation or spectral density function is of interest in this analysis. That is the one corresponding to the expected value of the product of the velocity components which are both parallel to the antenna axis. In consideration of this limitation and in the interest of simplified notation, the subscripts designating velocity component, correlation function, and spectral density will be omitted. Further, since the illuminated volume, at ranges of interest, is approximately a cylindrical section symmetrical about the antenna axis, it will be convenient to adopt the coordinate system illustrated in Fig. 5 (compare Fig. 2). Subscripts and coordinates are omitted here also.

If the velocity field is continuous in the illuminated volume and if, because hydrometeors are uniformly distributed, radar return is equally responsive to components of velocity throughout the volume, then the sample mean of velocity is
\[ m_v = \frac{1}{V} \int_V V(x, y, z) \, dV \] (67)

where $V$ is the volume of hydrometeors from which radar return is received at some instant of time (see [24]). The sample mean of velocity is responsible for the Doppler frequency defined in (7). The sample variance of velocity is
\[ \sigma_v^2 = \frac{1}{V} \int_V \left[ V(x_1, y_1, z_1) - \frac{1}{V} \int_V V(x_2, y_2, z_2) \, dV \right] \cdot \left[ V(x_1, y_1, z_1) - \frac{1}{V} \int_V V(x_3, y_3, z_3) \, dV \right] \, dV. \] (68)

The position coordinates of velocity bear subscripts to identify the dummy variables of each of the three integrations. The expected value of sample variance is, from (64) and (67) and after reversing the order of expectation and integration,
\[ \sigma^2 = E[\sigma_v^2] = R(0) - \frac{1}{V^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\xi) \, d\xi \, dV \] (69)

where by definition, the components of $\xi$ are
\[ \xi_1 = x_2 - x_1 \quad \xi_2 = y_2 - y_1 \quad \xi_3 = z_2 - z_1. \] (70)
Substitute (70) into (65) and the result into (69). Reverse the order of integration and transform to cylindrical coordinates

\[ x_i = x_i, \quad y_i = \rho_i \cos \phi_i, \quad z_i = \rho_i \sin \phi_i, \quad i = 1, 2 \]  

suggested by the symmetry of the illuminated volume. Let

\[ P = (\Omega_2^2 + \Omega_3^2)^{1/2} \Phi = \arctan(\Omega_3/\Omega_2) \quad \Omega_1 = \Omega_3 \]  

and find

\[
\sigma^2 = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{2\pi} \left[ 1 - \frac{\sin^2(\Omega_1 w/2)}{(\Omega_1 w/2)^2} \frac{J_1^2(\rho P)}{(\rho P/2)^2} \right] \\
\cdot S(\Omega_1, P, \Phi) d\Phi dP d\Omega_1.
\]  

If, in addition to being homogeneous, the velocity field is also isotropic, then the spectral density is not a function of \( \Phi \), and integration over \( 0 < \phi < 2\pi \) may be completed without further information. The term in brackets in (73) is a function of the dimensions of the illuminated volume. It limits \( \sigma^2 \) to contributions from the higher wavenumber part of the power spectrum.

\[
\rho_A(\sigma_\omega) = \sqrt{\frac{\pi}{2}} \frac{c}{2\sigma_\omega^w} \text{erf} \left( \frac{\sqrt{2}\sigma_\omega w}{c} \right) \\
\rho_B(\tau) = \exp \left[ -2 \left( \frac{\omega_0 GT}{c} \right)^2 \right] \\
\rho_C(V_a \sin \psi) = \exp \left[ -2 \left( \frac{\omega_0 \beta V_a \tau \sin \psi}{4c} \right) \right] \\
\rho_D(\psi) = \exp \left( -\frac{27\psi^2\tau^2}{8\beta^2} \right)
\]

CONCLUSIONS

The joint probability density function (63) of samples of weather radar return from hydrometeors at the same nominal range and scan angle was developed. It was shown in the evaluation of auto- and cross-correlation coefficients [(37), (38), (39), (40), and (44)] of mixer input phasor amplitudes, which appear as the sum of squares in the density function, that weather radar clutter is a function of the turbulent motion of hydrometeors; radar antenna beamwidth, scan angle, and scan rate; radar transmitter pulsewidth and frequency stability; and aircraft speed. If turbulence is inhomogeneous, the probability density function of the wind field and of hydrometeor velocity may be skewed from normal. Skewness, through coefficient \((1 + q^2)^{1/2}\), increases the effective value of correlation coefficient \( \rho \) in the joint probability density function (63). However, the influence of skewness measure \( \gamma_1 \) on the density function is insignificant if it is of the order found in the upper atmosphere.

Some appreciation for the relative importance of air turbulence and other parameters to the value of correlation coefficient \( \rho(\tau) \) in (49) can be gained by examining Fig. 6. The Gaussian and error function terms whose product is \( \rho(\tau) \) are plotted as a function of an appropriate variable. These terms are

\[
\rho_A(\sigma_\omega) = \sqrt{\frac{\pi}{2}} \frac{c}{2\sigma_\omega^w} \text{erf} \left( \frac{\sqrt{2}\sigma_\omega w}{c} \right) \\
\rho_B(\tau) = \exp \left[ -2 \left( \frac{\omega_0 GT}{c} \right)^2 \right] \\
\rho_C(V_a \sin \psi) = \exp \left[ -2 \left( \frac{\omega_0 \beta V_a \tau \sin \psi}{4c} \right) \right] \\
\rho_D(\psi) = \exp \left( -\frac{27\psi^2\tau^2}{8\beta^2} \right)
\]
boundaries are based upon measurements of a component of wind velocity transverse to the aircraft flight path. Approximate terms $\sin^2 (\alpha)(x^2)$ and $\sin^2 (\beta)(y^2 \beta^2)$ in (73) by Gaussian functions having the same first and second terms in the series expansions. Make the transformation

$$\Omega_1 = \Omega \sin \alpha \quad \Omega = \Omega \cos \alpha, \quad -\pi/2 \leq \alpha \leq \pi/2 \quad (76)$$

and integrate variables $\Phi$ and $\Omega$ over ranges: $0 \leq \Phi < 2\pi$ and $\Omega_e \leq \Omega < \infty$. $\Omega_e$ is the cutoff wavenumber, conventionally defined, of the term in brackets in (73). Expand the integrand in a binomial series convergent at either long or short range, and complete the integration over the range of $\alpha$ to find $\sigma^2$.

For example, at long range,

$$\sigma^2 = \frac{3C(\beta)^{2/3}}{2(2 \ln 2)^{1/3}} \sum_{i=0}^{\infty} \sin \frac{\pi x_i}{2(2i + 1)(2i + 3)}, \quad r^\beta > \sqrt{\frac{2}{3}}$$

$$\Omega_{10} = \Omega (\sin \alpha + \sin \beta) \quad \Omega = \Omega \cos \alpha, \quad -\pi/2 \leq \alpha \leq \pi/2 \quad (77)$$

where $C$ is the coefficient of $\Omega^{-5/3}$ in the Kolmogorov-Obukov equation, a function of turbulence class boundaries, and

$$a_0 = 1 \quad a_1 = \frac{1}{3} \left[ \left( \frac{2w}{r^\beta} \right)^2 - 1 \right] \quad a_2 = -\frac{1}{9} \left[ \left( \frac{2w}{r^\beta} \right)^2 - 1 \right]^2$$

$$a_i = -a_{i-1} \left( \frac{3i - 4}{3i} \right) \left[ \frac{2w}{r^\beta} \right]^2 - 1 \right], \quad \text{for} \ i \geq 3 \quad (78)$$

Finally, substitute $\sigma^2$ into the equation for $\rho_B$ and evaluate it as a function of $r$. The evaluations of $\rho_A$, $\rho_C$, and $\rho_D$ are straightforward.

On examination of Fig. 6, it is seen that the value of $\Phi(r)$ is determined primarily by air turbulence term $\rho_B$. Term $\rho_D$ is very nearly one at the minimum scan rate ($\psi = 0.42 \text{ rad/s}$) permitted by the Federal Aviation Agency for display of precipitation rate in a ±60° scan sector. The effect on $\rho(r)$ of cross-beam velocity in term $\rho_C$ is negligible if $\psi$ is suitably restricted. For example, $\rho_C$ is almost one at subsonic jet aircraft speeds if $| \psi | \leq 5^\circ$. $\rho_A$, the contribution of transmitter frequency jitter to $\rho(r)$, is significantly less than one. However, it too can easily be made almost equal to one by choosing a pulsewidth less than 3.5 µs. By using a coaxial magnetron in a well designed transmitter, $\sigma_\omega$ can be held to less than $10^{-5} \omega_0$. If pulsewidth is decreased, $\rho_B$ will be increased slightly at short range.

The accuracy of the mathematical model of radar return from turbulent hydrometeors can be improved by using a more realistic model of antenna pattern and by examining and refining some of the assumptions which model hydrometeor motion and distribution. Stackpole [25] has shown that, depending on the scale of turbulence and drop size, hydrometeor velocity lags wind velocity. Therefore, the spectrum of hydrometeor turbulence is not identical with that of atmospheric turbulence. Also, hydrometeors are probably not uniformly distributed over the antenna beam. Instead, the lateral extent of backscatter from hydrometeors is more reasonably determined by the dimensions and spatial distribution of rain cells in the antenna beam and by the distribution of rainfall rates in the rain cell. Some experimental work has been done to model hydrometeor spatial distributions [26], [27] but much more experimental data are required to accurately model hydrometeor distributions in the radar clutter model.

GLOSSARY OF SYMBOLS

$\alpha_k$ Phase change of radar return on reflection from the $k$th hydrometeor.

$\beta$ Beamwidth (radians).

$\gamma_1$ Skewness measure—the ratio of the third central moment to the cube of standard deviation.

$\Delta r_k, \Delta r_k(0)$ Range difference between the most distant hydrometeor illuminated and the $k$th hydrometeor (at $r = 0$).

$e', e$ Random phase of a signal transmitted by a noncoherent radar plus a constant if unprimed.

$\zeta$ Vector difference between velocity sample points.

$\eta$ Radar cross section per unit volume of hydrometeors.

$\Theta, \theta$ Mean polar angle, polar angle of a hydrometeor.

$\rho$ Wavenumber distance from reference parallel to the antenna axis to spectrum point in wavenumber space, $P = (\Omega_2^2 + \Omega_3^2)^{1/2}$.

$\rho_0(\tau, r), \rho_0(\tau, r)$ Correlation coefficient of signal samples at the same nominal range and scan angle.

$\rho_0$ Correlation coefficient of illuminated volume with range $r$ either a variable or a parameter.

$\Phi_0(\tau, r)$ Distance from antenna axis to velocity point, $\rho = (x^2 + y^2)^{1/2}$.

$\sigma_\omega^2$ Variance of the relative hydrometeor velocity parallel to the antenna axis.

$\tau$ Variance of pulse-to-pulse transmitted frequency difference.

$\Phi, \phi$ IP output power—phasor variance.

$\psi, \psi$ Time difference.

$\Omega$ Mean meridional angle, meridional angle of hydrometeors.

$\Phi, \phi$ Meridional angle of wavenumber $P$, of distance $\rho$.

$\psi, \psi$ Antenna scan angle or azimuth angle; scan rate (radians/second).

$\Omega_{n, \omega_0}$ Vector wavenumber, space frequency.

$\omega$ Angular frequency of the $n$th pulse transmitted, its mean value.
\(\omega_d, \omega_t\) Angular frequency of Doppler shift, of IF center.

c Velocity of propagation of electromagnetic radiation.

\(E(\cdot)\) Expected value of \(\{\cdot\}\).

\(e_k\) Amplitude of signal received from \(k\)th hydrometeor.

\(f(\cdot)\) Probability density function of \(\{\cdot\}\).

\(I_0(\cdot)\) Modified zero order Bessel function.

\(J_1(\cdot)\) Bessel function of the first kind and order one.

\(P(\cdot)\) Probability of event \(\{\cdot\}\).

\(q\) Coefficient of \(\rho\); a function of \(\gamma_t\).

\(R_{xy}(\tau)\) Correlation function of detector output \(y\).

\(R_{ij}(\ell)\) Correlation tensor of homogeneous velocity components \(V_i(r)\) and \(V_j(r + \ell)\).

\(r, r, r\) Range, range vector, unit range vector.

\(S_{ij}(\Omega), S(P, \Omega_1)\) Spectral density of velocity components \(V_i\) and \(V_j\), of velocity components parallel to the antenna axis.

\(t'\) Time since the last pulse transmitted.

\(t\) Time since the leading edge of the first of a number of consecutive pulses which illuminate the same hydrometeor.

\(t_p, t_r\) Pulswidth, pulse repetition period.

\(U(t)\) Mixer input signal.

\(u_1(t), u_2(t)\) Components of mixer input phasor amplitude.

\(V\) Relative hydrometeor velocity.

\(V, v\) Component of hydrometeor velocity parallel to the antenna and deviation from the mean of that component.

\(V_a\) Aircraft speed.

\(v_x, v_y, v_z\) \(x, y,\) or \(z\)-component of the deviation of relative hydrometeor (sometimes wind) velocity from the mean.

\(w\) Width of a volume from which backscatter is received at the same time.

\(w_1, w_2\) IF output phasor amplitudes.

\(X(t)\) IF amplifier output signal.

\(x_1(t), x_2(t)\) Components of IF output phasor amplitude.

\(y\) Detector output; envelope of IF output signal.

REFERENCES


